# Physics 4261: Homework 7 (due Mar. 6, 2017)

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7.1. **Density matrices and the Bloch vector**

In class we showed that

$$\langle \sigma_z \rangle = \mathrm{Tr}\ \sigma_z \rho = w,$$

where $u$, $v$, and $w$ define the density matrix

$$\rho = \frac{1}{2}\begin{pmatrix} 1+w & u+iv \\ u-iv & 1-w \end{pmatrix}$$

Note the following definitions of the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that $\langle \sigma_x \rangle = u$ and $\langle \sigma_y \rangle = -v$.

7.2. **Foot 7.1**

7.3. **Foot 7.2**

7.4. **Driven classical oscillator**

The middle section of this problem is extra credit, but the last part is required. Consider a (1D) classical system with mass $m$, charge $e$ and oscillator (angular) frequency $\omega$, with a driving field $E(t) = E\cos(\omega t)$.

(a) Write down the equations of motion. Yes, this is as easy as it sounds.

(b) Consider the case $\omega = \omega_0$. Show that $x(t) = \frac{eE}{2m\omega} t \sin(\omega t)$ is a solution with $x(0) = 0$, and $x'(0) = 0$.

(c) When $\omega \neq \omega_0$, show that

$$x(t) = \frac{eE}{m}\frac{1}{\omega_0^2 - \omega^2}\left(\cos(\omega t) - \cos(\omega_0 t)\right)$$

is a solution with $x(0) = 0$, and $x'(0) = 0$.

(d) Should the second solution converge to the first in the case $\omega \to \omega_0$?

(e) (Extra credit) Let's try to simplify the expression for the case $\omega \neq \omega_0$. Use the equation $\omega = \omega_0 - (\omega_0 - \omega)$ and the trigonometric formula $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$ to express the solution in terms of two components, one proportional to $\sin(\omega_0 t$, and the other proportional to $\cos(\omega_0 t)$.

(f) (Extra credit) Adding the amplitude in quadrature of each component, show that the oscillation is given by $x_0(t)\sin(\omega_0 t + \delta)$. Show that

$$x_0^2(t) = \frac{4e^2E^2}{m^2(\omega_0 - \omega)^2(\omega_0 + \omega)^2}\sin^2\{(\omega_0 - \omega)t/2\}.$$

Don't waste your time figuring out $\delta$, and remember that $A\sin(\omega_0 t) + B\cos(\omega_0 t) = \sqrt{A^2 + B^2}\sin(\omega_0 t + \tan^{-1}(B/A))$.

(g) (Extra credit) We are treating our atom as a harmonic oscillator. The frequency of the oscillator must be $\omega_0$, given by the energy spacing of the atom. But the mass of the oscillator is needed in the classical formula, and the bound electron might not have the same *effective* mass as the free electron. Use the fact that the energy for a displaced harmonic oscillator is $\frac{1}{2}m\omega_0^2 x_0^2$, the energy of a partially excited harmonic oscillator is $|c_2|^2\hbar\omega_0$ and the formula for the displacement in terms of the excitation fraction from class ($x_0 = 2|c_2||X_{12}|$) to derive an expression for the effective mass $m$.

(h) (Extra credit) Show that, if $\omega$ is close to $\omega_0$, so that $\omega_0 + \omega \approx 2\omega_0$, we can rearrange our formula into

$$x_0^2(t) = \frac{e^2E^2|X_{12}|^4}{\hbar^2}t^2\frac{\sin^2 x}{x^2},$$

and find $x$. You will need to substitute the mass from the previous part.

(i) In order to compare with an atomic system, we need to compute the excitation fraction. From class we derived that the $x_0 = 2|c_2|X_{12}$. Find an expression for $|c_2|^2(t)$. Compare your result with formula (7.16) from the book.