Physics 4261: Homework 9 Solutions

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9.1. Foot 8.2 Doppler broadening The two fine-structure components of the $2s$-$2p$ transition in a lithium atom (CVP Note: he refers here to $^7\text{Li}$) have wavelengths of 670.961 nm and 670.976 nm (in a vacuum).

(a) (CVP) What are the levels involved in these transitions?

The ground state is $2\,^2S_{1/2}$, and the two possible excited states are $2\,^2P_{3/2}$ and $2\,^2P_{1/2}$.

(b) (original) Estimate the Doppler broadening of this line in a room-temperature vapor.

Taking the formula $1.7u/\lambda$, with $u = \sqrt{2k_B T/m}$, I get 2.1 GHz.

(c) (original) Comment on the feasibility of observing the weak-field Zeeman effect in lithium. (Hint: What field would be needed to see the Zeeman effect? What defines the weak-field regime and how does this compare?)

In order to see Zeeman effect, the splittings would need to be of order 2.1 GHz. However, the fine structure splitting is itself only 10 GHz. Therefore, in order to get a big enough field to see Zeeman effect, we would be in intermediate-field rather than weak-field territory.

9.2. Foot 8.4 Hyperfine structure in laser spectroscopy

(a) What is the physical origin of the interaction that leads to hyperfine structure in atoms?

The coupling of the electron spin magnetic moment and the nuclear magnetic moment.

(b) Show that hyperfine splittings obey an interval rule which can be expressed as

$$\Delta E_{F,F-1} = A_{nlj} F,$$

i.e. the splitting of two sub-levels is proportional to the total angular momentum quantum number $F$ of the sub-level with larger $F$.

The splitting between $F$ and $F - 1$ levels is given by

$$2\Delta E = A_{\text{HF}} F(F + 1) - A_{\text{HF}} (F - 1) F,$$

$$\Delta E = A_{\text{HF}} F.$$
(c) The naturally-occurring isotope of cesium ($^{133}\text{Cs}$) has a nuclear spin of $I = 7/2$. Draw a diagram showing the hyperfine sub-levels, labelled by the appropriate quantum number(s), that arise form the $6^2S_{1/2}$ and $6^2P_{3/2}$ levels in cesium, and the allowed electric dipole transitions between them.

\[ \begin{array}{cccccc}
 & F' = 5 & & & & \\
 & F' = 4 & & & & \\
 & F' = 3 & & & & \\
 & F' = 2 & & & & \\
 & F = 4 & & & & \\
 & F = 3 & & & & \\
\end{array} \]

Figure 1: The Cs hyperfine structure with allowed electric dipole transitions.

(d) Explain the principle of Doppler-free saturation spectroscopy.

Two beams counter propagate and are only both on resonance for non-Doppler shifted atoms. When this occurs, the stronger pump beam saturates the atoms and the weak probe beam is attenuated less than it would otherwise be, leaving a small bump in the absorption profile.

(e) The figure shows the saturated absorption spectrum obtained from the $6^2S_{1/2}$-$6^2P_{3/2}$ transition in a vapor of atomic cesium, including the cross-over resonances which occur midway between all pairs of transitions whose frequency separation is less than the Doppler width. The relative positions of the saturated absorption peaks within each group are given below in MHz:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100.7</td>
<td>201.5</td>
<td>226.5</td>
<td>327.2</td>
<td>452.9</td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>75.8</td>
<td>151.5</td>
<td>176.5</td>
<td>252.2</td>
<td>353.0</td>
<td></td>
</tr>
</tbody>
</table>

Using these data and the information in the diagram, determine the extent to which the interval rule is obeyed in this case and deduce the hyperfine parameter $A_{nlj}$ for the $6^2S_{1/2}$ and $6^2P_{3/2}$ levels.

The crossovers are B, D, E, and b, d, e. The interval rule says that $(F - C)/(C - A) = 5/4$, but it is slightly off: 1.2476. Similarly, $(f - c)/(c - a) = 4/3$, but it is 1.3300. The
hyperfine constant of the excited state is \((F - C)/5 = 50.3\) MHz. The total hyperfine splitting of the upper levels is \((F - A) + (c - a) = 604.4\) MHz, so the ground state hyperfine splitting is this, plus the difference \((a - F)\), which is 9192.6 MHz.

(f) Estimate the temperature of the cesium vapor. (The wavelength of the transition is 852 nm.)

My estimation of the FWHM is something like \((f - a) = 353\) Mhz. At 300 K the broadening would be 386 MHz, so it’s close to room temperature.

9.3. **Foot 8.8 Convolution of Lorentzian line shapes**
A simple quantitative model of saturated absorption spectroscopy is given in Appendix D and this exercise examines some of the mathematical details.

(a) The convolution of two Lorentzian functions of equal width can be found using

\[
\int_{-\infty}^{\infty} \frac{1}{1 + (2y - x)^2} \frac{1}{1 + x^2} dx = \frac{\pi}{2(1 + y^2)}.
\]


Eqn. D6 is (call it B)

\[
B = f(v = 0) \frac{\Gamma^2/4}{x^2 + \Gamma^2/4} \times \frac{\Gamma^2/4}{2(\omega - \omega_0) - x + \Gamma^2/4} du,
\]

with \(u = 2x/\Gamma\). Continuing on we get

\[
B = \frac{f(v = 0) \Gamma}{k} \frac{\pi \Gamma}{4} \frac{1}{1 + 4(\omega - \omega_0)^2/\Gamma^2},
\]

\[
B = \frac{f(v = 0) \pi \Gamma^2}{k} \frac{1}{8} g_H(\omega).
\]

We need to compare to the non-saturated term, which involves

\[
A = f(v = 0) \frac{\Gamma^2/4}{x^2 + \Gamma^2/4} dx,
\]

\[
A = \frac{f(v = 0) \Gamma}{k} \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{u^2 + 1} du,
\]

\[
A = \frac{f(v = 0) \pi \Gamma}{k} \frac{1}{2}.
\]

Thus

\[
A - \frac{I}{I_{sat}} B = \frac{f(v = 0) \pi \Gamma}{k} \frac{1}{2} \left( 1 - \frac{I}{I_{sat}} \frac{\pi \Gamma}{4} g_H(\omega) \right),
\]

which is consist with D.7.
(b) The convolution of two Lorentzian functions of unequal widths is

$$\int_{-\infty}^{\infty} \frac{1}{a^2 + (y + x)^2} \frac{1}{b^2 + (y - x)^2} dx = \left( \frac{a + b}{ab} \right) \frac{\pi}{(2y)^2 + (a + b)^2}.$$  

Use this to show that taking into account the power broadening of the hole burnt in populations by the pump beam leads to a predicted line width in saturation spectroscopy of

$$\Gamma' = \frac{1}{2} \Gamma \left( 1 + \sqrt{1 + \frac{I}{I_{sat}}} \right).$$

In this case, we need to modify the first term to not assume small $I/I_{sat}$, that is

$$\frac{\Gamma^2/4}{x^2 + \Gamma^2/4} \to \frac{\Gamma^2/4}{x^2 + \Gamma^2/4(1 + I/I_{sat})}.$$  

Now, ignoring prefactors (as we only want linewidth), we take

$$a = \frac{\Gamma}{2}(1 + I/I_{sat}),$$
$$b = \Gamma/2,$$
$$y = \omega - \omega_0,$$

and we end up with

$$\frac{1}{4(\omega - \omega_0)^2 + \left[ \Gamma/2 + \Gamma/2(1 + I/I_{sat}) \right]^2} \propto \frac{1}{(\omega - \omega_0)^2 + \frac{1}{4} \left[ \Gamma/2 + \Gamma/2(1 + I/2I_{sat}) \right]^2},$$

which proves the linewidth $\Gamma' = \Gamma/2 + \Gamma/2(1 + I/I_{sat})$.