Physics 4261: Lectures for week 12

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12.1 3D cooling - Optical molasses

One point of the previous techniques is that they do not simply slow down atoms. The atoms are actually cooled. Pure slowing, for example, would take stopped atoms and push them backwards. But these techniques do not affect stopped atoms, because they are off-resonant until the very end! Therefore, you might ask if it is possible to cool atoms by similar methods, and the answer is in fact yes. For simplicity, let's start by thinking about cooling along just one axis (the z-axis).

Imagine now that we have two beams propagating in opposite directions. Then the force will be given by the scattering rate for beam 1, minus the scattering rate for beam 2 (because beam 2 points the other way). For atoms at rest, both beams are the same, and there is no force. However, let us calculate the force on a moving atom, assuming we are below saturation. The force from the left beam is

$$F_L = \frac{\hbar k \Gamma I}{2 I_{\text{sat}}} \frac{1}{1 + \frac{4(\delta_0 - kv)^2}{\Gamma^2}},$$

where δ_0 is the detuning seen by an atom at rest. Similarly, the entire force is

$$F = \frac{\hbar k \Gamma I}{2 I_{\text{sat}}} \left[\frac{1}{1 + \frac{4(\delta_0 - kv)^2}{\Gamma^2}} - \frac{1}{1 + \frac{4(\delta_0 + kv)^2}{\Gamma^2}} \right],$$

$$F = \frac{\hbar k \Gamma I}{2 I_{\text{sat}}} \frac{\frac{16 \delta_0 k v}{\Gamma^2}}{1 + \frac{8(\delta_0^2 + k^2 v^2)}{\Gamma^2} + \frac{16(\delta_0^2 - k^2 v^2)^2}{\Gamma^4}}.$$

I plot the shape on the board roughly. Let us consider the case that $v \to 0$. Then we can approximate this behavior as damping, $F = -\alpha v$, where

$$\alpha = \lim_{v \to 0} -\frac{8 \hbar k^2 \delta_0 I}{\Gamma I_{\text{sat}}} \frac{1}{1 + \frac{8(\delta_0^2 + k^2 v^2)}{\Gamma^2} + \frac{16(\delta_0^2 - k^2 v^2)^2}{\Gamma^4}},$$

$$\alpha = -\frac{8 \hbar k^2 \delta_0 I}{\Gamma I_{\text{sat}}} \frac{1}{\left[1 + (2\delta_0/\Gamma)^2\right]^2}.$$

So clearly, we want to choose $\delta_0 < 0$ in order to get actual damping, instead of unstable anti-damping.

12.1.1 Consequences of damping

Now let us consider a system subject to such a damping force. We are interested in the average kinectic energy (we are going to call this a "temperature", without worrying whether atoms subject to such cooling are in a proper thermal equilibrium). The kinetic energy (in z) is

$$\begin{aligned} E_z &= \frac{1}{2} m v_z^2, \\ \frac{dE_z}{dt} &= m v_z \frac{dv_z}{dt} = -\alpha v_z^2 = -\frac{2\alpha}{m} E_z. \end{aligned}$$

Thus, the energy is damped with timescale $\tau = m/(2\alpha)$. This is typically a few microseconds. If we put beams in all six directions, then the total kinetic energy is damped thusly as well.

12.1.2 Statistical fluctuations

The previous section considers a classical damping, in other words, the rate at which energy is removed from the system on average. But we really would like to calculate what happens when discrete photons are absorbed and emitted by the system. Consider at atom with momentum $\mathbf{p}_0$. Then a photon of momentum $\hbar\mathbf{k}$ is absorbed, and emitted in a random direction. The momentum space picture is now an atom with momentum lying somewhere on a sphere with radius $\delta p = \hbar k$ centered at $\mathbf{p}_0 - \hbar\mathbf{k}$. Now, expectation value of the change in energy will be

$$\begin{aligned} \langle \Delta E \rangle &= \frac{\langle (\mathbf{p}_0 - \hbar\mathbf{k} + \delta\mathbf{p})^2 \rangle}{2m} - \frac{p_0^2}{2m}, \\ \langle \Delta E \rangle &= \frac{\hbar^2 k^2 + \delta p^2 - 2\hbar \langle \mathbf{p_0} \cdot \mathbf{k} \rangle - 2\hbar \langle \delta\mathbf{p} \cdot \mathbf{k} \rangle + 2 \langle \mathbf{p}_0 \cdot \delta\mathbf{p} \rangle}{2m}, \\ \langle \Delta E \rangle &= \frac{2\hbar^2 k^2 - 2\hbar \langle \mathbf{p_0} \cdot \mathbf{k} \rangle}{2m}, \end{aligned}$$

The second term shows the effects of the average damping force we calculated above. The first term, on the other hand, is a total heating of $2E_r$ per photon scattered, where $E_r = \hbar^2 k^2/(2m)$. Now, our equation for the total heating/cooling rate is

$$\frac{dE}{dt} = 6 R_{\text{scat}} (2E_r) - 2\alpha E/m,$$

where the factor of six counts all the beams. Then

$$\begin{aligned} \frac{3}{2} k_B T &= \frac{6 m R_{\text{scat}} E_r}{\alpha}, \\ k_B T &= 4 m E_r \frac{R_{\text{scat}}}{\alpha}, \\ k_B T &= \frac{\frac{4\hbar^2 k^2}{2m} \frac{\Gamma}{2} \frac{I}{I_{\text{sat}}} \frac{1}{1 + (2\delta_0/\Gamma)^2}}{\frac{-8\hbar k^2}{\Gamma} \frac{I}{I_{\text{sat}}} \frac{1}{[1 + (2\delta_0/\Gamma)^2]^2}}, \\ k_B T &= \frac{-\hbar\Gamma}{8\delta_0} \left[1 + \left(\frac{2\delta_0}{\Gamma} \right)^2 \right]. \end{aligned}$$

Taking

$$k_B \frac{dT}{d\delta_0} = \frac{\hbar\Gamma}{8}\left[\frac{1}{\delta_0^2} - \frac{4}{\Gamma^2}\right] = 0,$$
$$\delta_{\text{opt}} = -\Gamma/2,$$
$$k_B T_D = \frac{\hbar\Gamma}{2}.$$

12.2 The MOT

Damping by itself leads to cooling, but it does not actually trap atoms. However, once again we can introduce Zeeman shifts in order to accomplish this (as with Zeeman slower). The idea is to circularly polarize the light, and use a magnetic field gradient. In this environment, we can ensure the polarization direction is always opposite to the magnetic field on the side with the beams facing in. This shifts the levels to bring red-detuned light closer to resonance, and therefore to add a restoring force element. Following the discussion above, we can add the additional element to the scattering formula (say in z direction)

$$F_L = \frac{\hbar k \Gamma I}{2I_{\text{sat}}} \frac{1}{1 + \frac{4(\delta_0 - kv - \beta z)^2}{\Gamma^2}},$$

where $\beta = \frac{g\mu_B}{\hbar} B_z z$. Clearly this contributes an additional confinement equal to $\alpha\beta/kz$,

$$F_z = -\alpha v_z - \frac{\alpha\beta}{k} z.$$

Note the restoring force is typically twice as strong in one direction as in the other two.

12.3 The dipole force

Here we are going to go through the derivation of the dipole force, which is the force coming from the refraction of the light beams by the atoms. [Draw picture of light rays bent by spheres]. Let us now proceed to a derivation. I'm going to take a sort of shortcut compared to the book. We are going to work in Born approximation, where the only force that acts is the electric field on the dipole moment of the atom.The first point is to note that with external field $\mathbf{E} = E_0 \cos(\omega t - kz)\hat{x}$, the dipole moment is proportional to

$$-e\mathbf{r} \propto A\hat{x}\left[u\cos(\omega t - kz) - v\sin(\omega t - kz)\right],$$

where u and v are from the optical Bloch equations. So note that u and v determine everything up to prefactors. Now, the interaction of the field with the dipole moment gives an interaction energy

$$U = A\left[u\cos(\omega t - kz_0) - v\sin(\omega t - kz_0)\right] E_0(x, y, z)\cos(\omega t - kz),$$

where we are allowing E_0 to have a large scale envelope (from the focus of a laser beam or some such). Now, to find the force, we take the gradient of the interaction potential. However, we have to be careful not to take the derivative of the atom's components (that's why they have subscripts).

Because the force the field provides is the gradient acting on the dipole moment. We can also write the force directly as

$$F_z = A\left[u\cos(\omega t - kz) - v\sin(\omega t - kz)\right]\left[\frac{\partial E_0}{\partial z}\cos(\omega t - kz) + E_0 k \sin(\omega t - kz)\right].$$

Taking the time average

$$\bar{F}_z = \frac{A}{2}\left[u\frac{\partial E_0}{\partial z} - vE_0 k\right].$$

The term proportional to vE_0k is the scattering force. The other term is the dipole force. Taking the ratio and comparing to previous results will give us the prefactor:

$$\begin{aligned}
F_{\text{dipole}}/F_{\text{scat}} &= \frac{-u\partial E_0/\partial z}{vE_0 k},\\
\frac{\partial E_0/\partial z}{E_0} &= \frac{1}{2}\frac{\partial I/\partial z}{I},\\
\frac{u}{v} &= \frac{2\delta}{\Gamma},\\
F_{\text{dipole}}/F_{\text{scat}} &= -\frac{\delta}{\Gamma k}\frac{\partial I/\partial z}{I},\\
\mathbf{F}_{\text{dipole}} &= \frac{\hbar\delta}{2}\frac{\nabla I/I_{\text{sat}}}{1 + 4\delta^2/\Gamma^2 + I/I_{\text{sat}}},
\end{aligned}$$

which allows us to define a potential (assuming we neglect the saturation intensity in the denominator)

$$\begin{aligned}
U_{\text{dipole}} &= \frac{\hbar\delta}{2}\frac{I/I_{\text{sat}}}{1 + 4\delta^2/\Gamma^2 + I/I_{\text{sat}}},\\
U_{\text{dipole}} &= \frac{\hbar\delta}{\Gamma}R_{\text{scat}} \approx \frac{\hbar\Gamma^2}{8\delta}\frac{I}{I_{\text{sat}}}.
\end{aligned}$$

12.4 Some basic Stat-Mech

Since Stat-Mech is not a prereq for the course, and since I do want to cover Bose-Einstein condensate (BEC) I will introduce briefly the concept of quantum statistics. The fundamental idea of statistical mechanics is that each state of a system which is allowed by conservation laws is equally probable. We then have two important conservation laws: conservation of energy and conservation of particle number. Let us take our model system to be a bath plus an isolated level, which we call the quantum state. Now the quantum state will be treated like a harmonic oscillator (if it is a boson; if it is a fermion, then it is treated as a two-level system). The state will have some energy E. Think of it like a particle in a box state (this concept will be more useful later as well). Therefore, adding a new particle will cause the energy of the quantum state to increase by E, and to conserve energy it will also cause the bath to decrease its energy by E, that is $U \to U - E$. Similarly, the number of particles in the bath will decrease by one for each particle added to the system, $N \to N - 1$. For

n particles in the quantum state we have $U \to U - nE$ and $N \to N - n$.

Now, there are many states in the bath. Probably many, many states, even if we limit to states with energy U and particle number N. We call this number Ω. In fact, the number of possible states will typically grow exponentially in the number of particles, because (for example) adding a new degree of freedom multiplies the number of possible states. Since we like to make quantities that are extensive (i.e. depend linearly on the number of particles), we will take the logarithm of the number of states, as call this the "entropy" $S = \ln \Omega$. So the likelihood of having one particle in the quantum state vs zero particles is

$$\frac{P(1)}{P(0)} = \frac{\Omega(U-E, N-1)}{\Omega(U, n)} = \exp\left\{-S(U, N) + S(U-E, n-1)\right\}.$$

Now, in general S will be some complicated quantity. But I am going to define the partial derivatives of S as follows:

$$\frac{\partial S}{\partial U} = \beta = \frac{1}{k_B T},$$
$$\frac{\partial S}{\partial N} = -\beta\mu = -\frac{\mu}{k_B T},$$

where T is the temperature, and μ is the chemical potential. If we were spending more time on thermo or stat-mech, we'd go into this, but for now just trust me that this is how temperature is defined. Now the ratio of probabilities is

$$\frac{P(1)}{P(0)} = e^{-(E-\mu)/k_B T}.$$

This is called the Gibbs factor, and if μ is set to zero it is called the Boltzmann factor. Then up to a constant

$$P(n) = \frac{1}{\mathcal{Z}} e^{-n(E-\mu)/k_B T},$$

Finding ratios of probabilities is great, but to get the actual probabilities, we need to compute the normalization factor

$$\mathcal{Z} = \sum_{n=0}^{N} e^{-n(E-\mu)/k_B T}.$$

This is called the grand partition function. For fermions, we have only two allowed levels, occupied and unoccupied, so

$$\mathcal{Z} = 1 + e^{-(E-\mu)/k_B T},$$
$$P(1) = \frac{e^{-(E-\mu)/k_B T}}{1 + e^{-(E-\mu)/k_B T}},$$
$$P(1) = \langle n \rangle = \frac{1}{e^{(E-\mu)/k_B T} + 1}.$$

This is called the Fermi-Dirac distribution. It specifies the equilibrium occupation level for non-interacting fermions. To get the total number of fermions, we would sum (integrate) this over all energy levels (say, using the particle-in-a-box levels and taking the limit as the box becomes infinite). We will do this in a minute, but since we are talking about BEC, we are going to move to bosons. For bosons

$$\mathcal{Z} = \sum_{n=0}^{N} e^{-n(E-\mu)/k_B T} = \frac{1}{1 - e^{-(E-\mu)/k_B T}} = \frac{1}{1 - e^{x}},$$
$$\langle n \rangle = \frac{1}{\mathcal{Z}} \sum_{n=0}^{N} n e^{-n(E-\mu)/k_B T} = \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial x},$$
$$\langle n \rangle = \frac{e^{x}(1 - e^{x})}{(1 - e^{x})^2} = \frac{1}{e^{(E-\mu)/k_B T} - 1}.$$

This is the Bose-Einstein distribution.