Physics 4261: Lectures for week 14

Prof. Colin V. Parker (cparker@gatech.edu)
Georgia Institute of Technology

14.1 Non-optical trapping

We've focused a lot on interactions with light because they are very important in modern atomic physics, but other methods are used as well, particularly DC (or quasi-DC) electric and magnetic fields. Here we will talk about the possible kinds of traps that can be made at DC.

14.1.1 Earnshaw's theorem

A major result in E&M is Earnshaw's theorem, which says that you cannot have an empty region of space which is a local maximum (or minimum) of the potential. In particular, let us state the following. If a particular region of space is a local maximum, then the electric field will point inward along a surface bounding a small neighborhood of the maximum. Then, by Gauss' law

$$-\oint \mathbf{E}\cdot\mathbf{dA} = \int \nabla^2\phi\, dV = -\frac{\rho}{\epsilon_0} < 0.$$

So if a potential has a local maximum, then the Laplacian is negative in some neighborhood around the maximum, and since the Laplacian can only be zero, no local maxima are possible. The same argument restricts local minima, because $-\nabla^2\phi$ is also non-negative However, we can have dipoles. For a dipole, say, a magnetic dipole, the atom will feel an effective potential $V = -\boldsymbol{\mu}\cdot\mathbf{B}$. If the atom moves slowly (adiabatically), it will stay in the same m_z state even as the field changes axis, and I only care about the absolute value $|\mathbf{B}| = B$, so the potential is

$$V = g_F\mu_B M_F B.$$

Note that M_F can be of either sign, and that at large field this approximation breaks down. Now the basic question to ask is, can a local maximum exist in the quantity B, or equivalently B^2, and the answer is no, although a local minimum can. Thus states with $m_F > 0$ are trappable, and known as high-field seekers. Let's prove this easy E&M result. We only need to prove that the Laplacian of $|B|^2$ is non-negative.

$$\nabla^2|\mathbf{B}\cdot\mathbf{B}| = \partial_i\partial_i B_j B_j,$$

using Einstein notation.

$$\partial_i\partial_i B_j B_j = 2\partial_i B_j \partial_i B_j = 2\underbrace{(\partial_i B_j)(\partial_i B_j)}_{\text{non-negative}} + 2\underbrace{B_j\partial_i\partial_i B_j}_{\text{secretly zero}}.$$

So it remains to prove that the second term is negative. Let us first prove

$$B_i \partial_i \partial_j B_j = B_i \partial_i (\nabla \cdot \mathbf{B}) = 0.$$

Then

$$\begin{aligned} B_j \partial_i \partial_i B_j &= B_j \partial_i \partial_i B_j - B_i \partial_i \partial_j B_j = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) B_m \partial_i \partial_l B_j, \\ &= \epsilon_{ijk}\epsilon_{klm} B_m \partial_i \partial_l B_j = -\mathbf{B} \cdot (\nabla \times (\nabla \times \mathbf{B})) = 0. \end{aligned}$$

The book goes through a little bit more detail on how to make local minima of $|B|$, but the easiest way is to choose the scalar magnetic potential

$$\begin{aligned} \phi_B &= z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2, \\ B &= -\frac{B_0}{2}x\hat{x} - \frac{B_0}{2}y\hat{y} + B_0 z\hat{z}, \end{aligned}$$

which is clearly zero at the origin, and nowhere else. The field magnitude is

$$B_0\sqrt{4z^2 + x^2 + y^2}.$$

This is called the quadrupole trap, and it only has the disadvantage that the motion will eventually lead to non-adiabatic behavior near the field zero. Taking the magnetic moment to be approximately the Bohr magneton, we get a trapping potential of $\mu_B/k_B = 0.67$K/T. For a 10 T/m gradient, the force is of order 10^{-22} N.

14.2 Penning traps

The final kind of trap we are going to consider is the Penning trap. In a Penning trap, we have both electric and magnetic fields. The fields are given by

$$\begin{aligned} \mathbf{E} &= \frac{E}{r_0}x\hat{x} + \frac{E}{r_0}y\hat{y} - 2\frac{E}{r_0}z\hat{z}, \\ \mathbf{B} &= B\hat{z}. \end{aligned}$$

The idea is that the electric field gives trapping in the vertical direction, but repels in the plane. However, the magnetic field will bring the electron back around and prevent it from escaping! Here are the equations of motion (in x and y - the equations of motion in z are SHO)

$$\begin{aligned} \ddot{x} - \frac{qB}{M}\dot{y} - \frac{qE}{mr_0}x &= 0, \\ \ddot{y} + \frac{qB}{M}\dot{x} - \frac{qE}{mr_0}y &= 0. \end{aligned}$$

Letting $\omega_c = qB/M$ be the cyclotron frequency and $\omega_0 = E/Br_0$ be the another frequency, we have

$$\begin{aligned} \ddot{x} - \omega_c\dot{y} - \omega_c\omega_0 x &= 0, \\ \ddot{y} + \omega_c\dot{x} - \omega_c\omega_0 y &= 0. \end{aligned}$$

We now guess the solution (based on simple cyclotron motion)

$$x = \cos(\omega t), \qquad\qquad y = -\sin(\omega t),$$

leading to

$$-\omega^2 + \omega_c \omega - \omega_c \omega_0 = 0.$$

This has solutions given by

$$\omega = \frac{\omega_c}{2} \pm \sqrt{\frac{\omega_c^2}{4} - \omega_c \omega_0},$$
$$\omega = \frac{\omega_c}{2} \left(1 \pm \sqrt{1 - \frac{4\omega_0}{\omega_c}} \right).$$

For $\omega_0 \ll \omega_c$, this simplifies down into $\omega = \omega_c, \omega_0$. The slow angular frequency is called the magnetron frequency, $\omega_m \approx \omega_0$, where this assumes $\omega_0 \ll \omega_c$. There is one more frequency to calculate, in the z-direction, where

$$\ddot{z} + \frac{2qE}{mr_0} = 0,$$
$$\ddot{z} + 2\omega_c \omega_0 = 0,$$
$$\omega_z = \sqrt{2\omega_c \omega_0}.$$

Usually, the hierarchy goes $\omega_m \ll \omega_z \ll \omega_c$.