

Physics 4261: Lectures for week 7

Prof. Colin V. Parker (cparker@gatech.edu)
Georgia Institute of Technology

In the next unit we are going to focus on the interaction of atoms with fields. This is of fundamental importance to atomic physics, because fields (particularly light) play a huge role in experimental atomic physics, and light, together with atoms, is one of the most “quantum” phenomena that we know. We have already seen how to understand the Zeeman effect, or the coupling of atoms (and the associated level splitting) to DC magnetic fields. We will not focus on the Stark effect, the analogous phenomenon for electric fields, because it has limited uses, and we will focus directly on AC electromagnetic fields, particularly light. However, some intuition can be gained by taking the zero-frequency limit of the AC fields.

7.1 Light scattering by atoms

Up until now, we have only treated the interaction with light as a way to measure the atomic level spacings and energies. Now we are going to ask the question, what happens when I turn on light at a specific wavelength? How does the intensity of light matter? How close to the exact wavelength do I need to be? For this we need a mathematical framework for light scattering, which will ultimately be the Optical Bloch Equations (OBE).

The OBE are a quite general set of equations applicable in a broad range of cases (intense light and weak light, on resonance, off resonance, etc.) We will derive them in two ways:

First, we return to our argument about mixed states and time dependent moments:

1. We calculate the time-dependent moment for an atom in a superposition of two states, with the majority in one state (this was problem 2.10 from the book).
2. In this limit, I can consider the system to be similar to a harmonic oscillator, where one state is the ground state of the oscillator and the other is the first excited state. A *small* fraction of excited state makes this similar to a so-called “coherent state”, which is the ground state displaced from the center of the potential (a true harmonic oscillator coherent state also involves second and higher excited states). But take the limit of low excitation fraction, and I have something like a harmonic oscillator.
3. I know how a dipole harmonic oscillator should radiate classically. Because of the magic of the harmonic oscillator, the expectation value obeys classical equations of motion.
4. Therefore, I know the expected power radiated P and therefore the average scattering rate A_{21} of the quantum system.

5. This allows me to write down a set of equations for the time evolution of the system *for small* α . However, these cannot be traditional quantum equations, because the system is losing energy! Therefore we will introduce a set of so-called Lindblad equations, which operate on density matrices instead of wavefunctions.
6. Finally, from the form of the equations, we will see that, within linear quantum mechanics, only a small set of possible damping terms are allowed. Therefore, we can extend our argument to *all* values of α ! This would confuse the classical physicist, who would assume that there was no power radiated when the system is in the pure excited state.

The second way:

1. We are going to derive the equations for a classical (and again for a quantum) oscillator which is not radiating, but instead absorbs energy from an oscillating field, this follows 7.1.1 and 7.1.2 from the book.
2. Then we can make an argument following Einstein using some basic thermodynamics to deduce the spontaneous transition rate, this follows 7.2.

7.2 Density matrix

The density matrix of a system in a pure state $|\psi\rangle$ is given by $\rho = |\psi\rangle \langle\psi|$. A few properties are immediately obvious in this form:

1. ρ is Hermitian.
2. $|\psi\rangle$ is an eigenvector of ρ with eigenvalue 1.
3. The Hilbert space orthogonal to ψ constitutes an eigenspace with eigenvalue 0.
4. $\text{Tr } \rho = 1$.
5. $\langle\psi' | \rho | \psi'\rangle \geq 0$ for all $|\psi'\rangle$. Thus ρ is “positive semi-definite”.

If we now define a more general density matrix as a sum over different pure states with classical weights $\rho = \sum_i c_i \rho_i$, with $\sum_i c_i = 1$, then properties 1, 4, and 5 still apply. Let’s focus specifically on the two by two case as we did above. Write the density matrix as

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}.$$

The diagonal elements represent the classical probability to be in a given state, and the off diagonal elements are called “coherences”, when they are at their maximum the state is called coherent (i.e. it is a pure quantum state). When the coherences are zero the state is in a classical mixture of the two states. Hermiticity shows us that the diagonal elements should be real, and that $\rho_{21} = \rho_{12}^*$. Furthermore, we know that $\rho_{11} + \rho_{22} = 1$. Thus, the most general two by two density matrix can be written in terms of three real quantities:

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + w & u + iv \\ u - iv & 1 - w \end{pmatrix}.$$

Now, this matrix is Hermitian and has trace one. But is it positive semi-definite? Let us see. The eigenvalues are:

$$\lambda = 1 \pm \sqrt{u^2 + v^2 + w^2}.$$

This means we must have

$$u^2 + v^2 + w^2 \leq 1,$$

which leads to a convenient geometric representation of the density matrix as a point inside the unit sphere in \mathbb{R}^3 . This is known as the “Bloch” sphere. What’s even cooler is that the geometric representation is not even totally abstract, in the case of spin-half systems! The expectation value of a quantity, say the Pauli matrix σ_z is given by

$$\langle \sigma_z \rangle = \text{Tr} \sigma_z \rho = \frac{1}{2} \text{Tr} \begin{pmatrix} 1 + w & u + iv \\ -u + iv & -1 + w \end{pmatrix} = w.$$

Can you guess what $\langle \sigma_x \rangle$ is? In your homework you will prove that it is u . But in general, it is ok to think of the Bloch sphere as being abstract. Another useful concept is the length of the Bloch vector, which tells us about the coherence of the state. If it is zero, we have no knowledge of the state at all, if it is one then we are in a pure state. This can be related to entropy as well.